ON A RIGID-PLASTIC ANNULAR PLATE UNDER IMPULSIVE LOAD

(UDAR PO KOL'TSEVOI ZHESTKO-PLASTICHESKOI PLASTINKE)

PMM Vol.23, No.1, 1959, pp.172-175

G.S. SHAPIRO
(Moscow)

(Received 3 November 1958)

Gvozdev [1] was apparently the first to propose an idea for utilizing the rigid-plastic analysis for the investigation of the dynamic behavior of plates. He used this idea in connection with the strength determination of rectangular plates under the action of explosive loads. (A more detailed bibliogaphy can be found in the review [2] by Rakhmatulin and the author*).

The theory of the axisymmetrical dynamic bending of rigid-plastic circular plates was established by Hopkins and Prager [3]. They considered a freely-supported circular plate under the action of a uniformly distributed load which was kept constant for some period of time and then suddenly removed. Wang and Hopkins [4] analyzed plastic deformations of a circular built-in plate, all points of which were given the same velocity at an initial instant. Both solutions pertain to continuous plates. Nothing has yet been published on the annuli.

Nor are there any publications on experimental verification of the theory of the dynamic bending of rigid-plastic plates. This is apparently due to the difficulties encountered in setting up experiments to correspond exactly with the cases for which theoretical solutions are found. For beams, however, experimental studies conducted by Parkes [5] have verified a series of theoretical solutions.

---

* The problem of an annular plate under the action of a constant load and in its simple formulation, viz., assuming that the static deformations of the plate are preserved during the motion (which is characterised by the forming of a stationary hinge circle along the clamped edge and by the transition of a flat plate into a truncated cone), was treated at MGU [Moscow State University] by Tsimbal and Mitskevich.

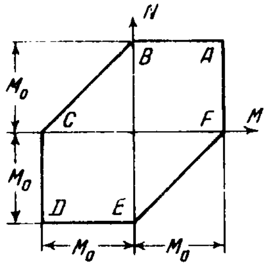

Fig. 1.

A simple solution of a problem which can be verified experimentally is given in this paper. Consider a thin plate in the shape of an annulus which is clamped along its inner edge $r = a$ and free along its outer edge $r = R$. The outer edge is given a constant velocity v_0, which at some later time $t = T$ is suddenly removed. The subsequent motion of the plate consists of three phases. During the first phase, $0 < t < t^+$, the hinge circle which appears at the initial moment along the contour to which the impulse was applied is moving towards the center of the plate and reaches the clamped edge at time t^+. It is assumed that $t^+ < T$. (The case when $t^+ > T$ can be considered analogously.) The second phase is characterized by the formulation of a stationary hinge along the clamped edge, $r = a$. The plate then rotates uniformly about this hinge, and no inertia forces are present. The third phase starts after the cessation of the applied velocity v_0 along the free edge, $r = R$. The plate proceeds to rotate about the same hinge along the clamped edge $r = a$. The duration of each phase of the motion, the deformation of the plate and the distribution of the shear forces and bending moments are determined in this paper.

1. Basic considerations. It is assumed that the plate is made of a rigid-ideally plastic material which obeys Tresca's yield condition and the associated flow rule. Let us denote the bending moments in the radial and tangential directions by M and N respectively. The yield locus in the MN plane for this case is a hexagon $ABCDEF$, Fig. 1.

At the initial moment, $t = 0$, the plate is in the equilibrium position (Fig. 2*a*). Thus, the initial conditions for the deflections are

$$w(r, 0) = w_t(r, 0) = 0 \tag{1.1}$$

At the time $t = 0$, the contour $r = R$ is displaced in the axial direction with a given constant velocity v_0 during the interval $0 < t < T$.

It is necessary for the solution of the problem to assume a certain deformation velocity field. Dynamic solutions of the plate problems frequently start with an assumption of the velocity field which corresponds to the static problem of the limiting state of the plate. Obviously this approach is not applicable in the given case, for when a limiting ringwise pressure is acting along the contour $r = R$, the hinge is formed along the edge $r = a$. The plate starts to rotate about this hinge and will change its shape from a flat to a truncated cone. If we assume the existence of a similar velocity field in the dynamic problem, then, the rotation about the contour $r = a$ being uniform, there will be no inertia forces, and we come to the absurd conclusion that no matter what the magnitude of the velocity v_0 is, the strength of the plate (the shear force at $r = R$) will remain constant and equal to the limiting static strength. Thus, some other velocity field has to be accepted.

Let us assume that along the contour $r = R$ at the initial moment $t = 0$, there appears a non-stationary hinge circle, the radius ρ of which is decreasing. At the end of the first phase of the motion $\rho = a$. For convenience the time for each phase is counted from zero. The times will be denoted by t_1, t_2, t_3 for the first, second and third phase respectively. Let us introduce the following dimensionless magnitudes

$$\alpha = \frac{a}{R}, \quad \eta = \frac{r}{R}, \quad \xi = \frac{\rho}{R}, \quad \beta = \frac{(\mu v_0 R^2)}{12 M_0}, \quad \tau_i = \frac{t_i}{\beta} \quad (i=1,2,3)$$
$$m = \frac{M}{M_0}, \quad n = \frac{N}{M_0}, \quad q = \frac{QR}{M_0}, \quad z_i = \frac{w_i}{v_0 \beta} \quad (i=1,2,3) \tag{1.2}$$
$$\dot{k}_r = \frac{R^2}{v_0}\chi_r, \quad \chi_r = -\frac{\partial^2 \dot{z}}{\partial \eta^2}, \quad k_\varphi = \frac{R^2}{v_0}\chi_\varphi, \quad \chi_\varphi = -\frac{1}{\eta}\frac{\partial z}{\partial \eta}$$

where μ is the surface density of the plate, Q the shear force and k_r and k_ϕ the curvature rates in the radial and tangential directions respectively.

The dot indicates differentiation with respect to t_i or τ_i, z denotes the total bending, $z = z_1$ for the first phase (also $t = t_1$), $z = z_1 + z_2$ for the second and $z = z_1 + z_2 + z_3$ for the third.

2. The first phase of the motion. Assume that during the first phase the velocity field is determined by the following relationship

$$\dot{z} = \frac{\eta - \xi}{1 - \xi} \quad (\xi \leqslant \eta \leqslant 1), \qquad \dot{z} = 0 \quad (\alpha \leqslant \eta \leqslant \xi) \tag{2.1}$$

For the accelerations we obtain

$$\ddot{z} = -\frac{\xi(1-\eta)}{(1-\xi)^2} \quad (\xi \leqslant \eta \leqslant 1), \qquad \ddot{z} = 0 \quad (\alpha \leqslant \eta \leqslant \xi) \tag{2.2}$$

Parkes [5] used similar representation for a somewhat different problem, viz., for a beam subjected to the impact of a finite mass.

The distribution of the velocities and accelerations is shown in Fig.2b and Fig. 2c. From (1.2) and (2.1) we find the rates of change of the curvature

$$\dot{\chi}_r = 0, \quad \dot{\chi}_\varphi = -\frac{1}{\eta}\frac{1}{1-\xi} \quad (\xi \leqslant \eta \leqslant 1), \qquad \dot{\chi}_r = 0, \quad \dot{\chi}_\varphi = 0 \quad (\alpha \leqslant \eta \leqslant \xi) \tag{2.3}$$

From (2.3) we conclude that for $\xi \leqslant \eta \leqslant 1$ the curvature velocity vector is orthogonal to the side *DE* of the yield locus. To the circle $\eta = 1$ corresponds regime *E*, and to the non-stationary hinge circle $\eta = \xi$ corresponds regime *D*. The annulus $a < \eta < \xi$ is also in regime *D*.

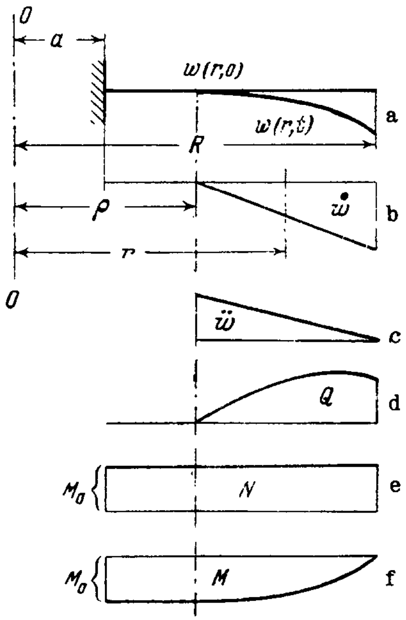

Fig. 2.

Hence it follows that in this region

$$q \equiv 0 \tag{2.4}$$

The equation of the dynamic equilibrium which takes (2.4) into account can be written as follows

$$\frac{d(m\eta)}{d\eta} - n = \eta q = \int_\xi^\eta \ddot{z}\eta \, d\eta \tag{2.5}$$

Noticing that $n = -1$, and using (2.2) we get

$$\frac{d(m\eta)}{d\eta} + 1 = \frac{\dot{\xi}}{(1-\xi)^2}\int_\xi^\eta (1-\eta)\,\eta d\eta \tag{2.6}$$

Integrating this equation between the limits $m = -1$ for $\eta = \xi$ and

$m = 0$ for $\eta = 1$, we find the velocity of the motion of the hinge circle,

$$\dot{\xi} = -[(1-\xi)(1+3\xi)]^{-1} \tag{2.7}$$

It follows that

$$\tau_1 = \xi^3 - \xi^2 - \xi + c$$

Where c is an arbitrary constant. This constant can be determined from the condition that $\xi = 1$ for $\tau_1 = 0$. Thus, $c = 1$. Hence,

$$\tau_1 = (1-\xi)(1-\xi^2) \tag{2.8}$$

The instant τ_1^+, the end of the first phase, can be found by letting $\xi = \alpha$. Thus we have

$$\tau_1^* = (1-\alpha)(1-\alpha^2) \tag{2.9}$$

From (2.2), (2.5) and (2.7) it is possible to determine the shear force

$$q = -2[(1-\xi)^3(1+3\xi)]^{-1}\left[3\left(\eta - \frac{\xi^2}{\eta}\right) - 2\left(\eta^2 - \frac{\xi^3}{\eta}\right)\right] \tag{2.10}$$

Along the edge of application of the impulse we have $\eta = 1$

$$q = -2(1+2\xi)[(1-\xi)(1+3\xi)]^{-1} \tag{2.11}$$

At the initial instant of the action of the impulse $\xi = 1$ and shear force has a singularity on the circle $\eta = 1$. This is to be expected, since otherwise for $\eta = 1$ the radial moment could not reach its limiting value $m = -1$. The shear stress is zero along the clamped edge $\eta = \alpha$ during the whole first phase of the motion. Radial bending moments are determined with help of the equalities (2.5) and (2.7). We obtain

$$m = -1 + [(1-\xi)^3(1+3\xi)]^{-1}[\eta^2(2-\eta) + \eta^{-1}(4-3\xi)\xi^3 + 2\xi^2(2\xi-3)] \tag{2.12}$$

It is easy to verify that for $\eta = \xi$ the moment m reaches an extremum, $m = -1$. Thus, the yield stress is nowhere exceeded.

For the determination of the deflection z_1, we can employ several different procedures. First of all ξ can be eliminated from (2.1) by means of (2.8), and then (2.1) can be directly integrated with respect to τ_1. However, it is simpler to eliminate the time τ_1 from (2.1) and (2.8) and then to integrate with respect to ξ. We get

$$dz = -(\eta - \xi)(1+3\xi)\,d\xi \tag{2.13}$$

Hence it follows

$$z = -\int_\eta^\xi (\eta-\xi)(1+3\xi)\,d\xi = \frac{1}{2}(\eta-\xi)^2(1+2\xi+\eta) \tag{2.14}$$

An alternative method for determination of the deflection was shown by Hopkins and Prager [3]. Since along the hinge circle $\xi = \xi(\tau)$, there exists a jump

$$\left[\frac{\partial \dot{z}}{\partial y}\right] = (1-\xi)^{-1} \tag{2.15}$$

and since it is known [3] that

$$\left[\frac{\partial \dot{z}}{\partial \eta}\right] + \dot{\xi}\left[\frac{\partial^2 z}{\partial \eta^2}\right] = 0 \tag{2.16}$$

Then, taking (2.7) into account, it follows that

$$\left[\frac{\partial^2 z}{\partial \eta^2}\right] = 1 + 3\xi \tag{2.17}$$

Since the hinge circle is moving into an undisturbed region inside which $\partial^2 z/\partial \eta^2 = 0$, the following must be satisfied

$$\left[\frac{\partial^2 z}{\partial \eta^2}\right] = \frac{\partial^2 z}{\partial \eta^2} \tag{2.18}$$

Taking into account that in the disturbed region $\xi < \eta < 1$ according to (2.1), $\partial^2 z/\partial \eta^2 = 0$, we have

$$\frac{\partial^2 z}{\partial \eta^2} = 1 + 3\eta \tag{2.19}$$

Integrating (2.19) with the boundary conditions $\partial z(\xi, \tau)/\partial \eta = z(\xi, \eta) = 0$, we again obtain (2.14). Putting $\xi = a$ in (2.14) we get the deflections $z = z^+$ at the end of the first phase.

The diagrams of the distribution of Q, M and N along the radius r are shown in Figs. 2*d*, 2*e*, and 2*f*.

3. Second phase. In the second phase the plate is rotating uniformly about the clamped circle $\eta = a$. The displacements z_2 in this phase are determined from

$$z_2 = \frac{\eta - \alpha}{1 - \alpha}\tau_2 \tag{3.1}$$

Let us denote the duration of the second phase by τ_2^+, and let $k = \tau_2^+/\tau_1^+$. Thus, at the end of the second phase the deflection $z_1 = z_2^+$ is

$$z_2^* = k(\eta - \alpha)(1 - \alpha^2)$$

The shear forces and the bending moments in the second phase are the same as for the limiting static loading, [6].

4. Third phase. The third phase is characterized by the rotation of the plate, due to inertia, about the stationary hinge circle $\eta = a$. The deflections $z_3(\eta, \tau_3)$ are sought in the following form:

$$z(\eta, \tau_3) = Z(\tau_3)\frac{\eta - \alpha}{1 - \alpha}$$

where $Z(\tau_3)$ is an unknown function of time. To determinate this function we utilize the dynamic equilibrium equation, which in this case we write as follows:

$$\frac{d(\eta m)}{d\eta} - n = -\int_\eta^1 \ddot{z}_3 \eta d\eta \tag{3.2}$$

Here we have allowed for the fact that $q = 0$ for $\eta = 1$. Using the arguments which were applied for the first phase, it follows that $n = -1$. Substituting (3.1) into (3.2) and integrating we find

$$\frac{d(\eta m)}{d\eta} + 1 = -\frac{Z}{1-\alpha}\left(\frac{1}{3} - \frac{\eta^3}{3} - \frac{\alpha}{2} + \frac{\alpha\eta^2}{2}\right) \tag{3.3}$$

Taking into account that $m(\alpha, \tau_3) = m(1, \tau_3) = 0$, we have $z = -(3 - 5\alpha + \alpha^2 + \alpha^3)^{-1}$. Noticing that $z(0) = 1$ and $z(0) = 0$, we get

$$Z = \tau_3 - \frac{1}{2}(3 - 5\alpha + \alpha^2 + \alpha^3)^{-1}\tau_3^2 \tag{3.4}$$

The plate will stop rotating at the instant $\tau_3^+ = 3 - 5\alpha + \alpha^2 + \alpha^3$, and the total stopping time $\tau^+ = \tau_1^+ + \tau_2^+ + \tau_3^+$. The magnitude of the total deflection at the instant $\tau = \tau^+$ is

$$z^* = z_1^* + z_2^* + z_3^* = \frac{1}{2}(\eta - \alpha)[\eta(\eta + \alpha + 1) + 2k(1 - \alpha^2) + 3(1 - \alpha - \alpha^2)] \tag{3.5}$$

BIBLIOGRAPHY

1. Gvozdev, A.A., K raschetu konstruktsii na deistvie vzryvnoi volny (On the calculation of constructions under the action of an explosive wave). *Stroitelnaya promyshlennost* No. 1-2, 1943.

2. Rakhmatulin, Kh.A. and Shapiro, G.S., Rasprostranenie vozmushchenii v nelineino-uprugoi srede (Propagation of a turbulence in nonlinear elastic medium). *Izvestia AN SSSR, OTN* No. 2, 1955, pp. 68-89.

3. Hopkins, H.G. and Prager, W., Dinamika plasticheskoi krugloi plastinki (Dynamics of a plastic circular plate). *Mekhanika, sb. perevodov i obzorov IL* No. 3, 1955, pp. 112-122.

4. Wang, A.J. and Hopkins, H.G., O plasticheskoi deformatsii zadelannoi po kraiu krugloi plastinki (On the dynamic deformations of a built-in circular plate). *Mekhanika, sb. perevodov i obzorov IL* No. 3, 1955, pp. 123-138.

5. Parkes, E.W., Some simple experiments on the dynamic plastic behavior of mild-steel beams. *British Welding Journal*, August 1956, pp. 362-366.

6. Iliushin, A.A., *Plastichnost (Plasticity)*. ONTI, 1948.

Translated by R.M.E.-I.